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On the existence of solutions of variational inequalities in Banach spaces

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Abstract

In this paper, we study the existence of the solution of the variational inequality $\langle Tx - \xi, y - x \rangle \geq 0$ by applying the generalized projection operator $\pi_K : B^* \rightarrow B$, where B is a Banach space with dual space B^* and by using the well-known FanKKM Theorem.

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1. Introduction

Let B be a Banach space with dual space B^* . As usually, $\langle \phi, x \rangle$ denotes the duality pairing of B^* and B , where $\phi \in B^*$ and $x \in B$. (If B is a Hilbert space, $\langle \phi, x \rangle$ denotes an inner product in it.) Let K be a nonempty, closed and convex subset of B and $T : K \rightarrow B^*$ a mapping. Let us consider the following variational inequality:

$$\langle Tx - \xi, y - x \rangle \geq 0, \quad \text{for every } y \in K, \quad (1.1)$$

where $\xi \in B^*$. An element $x^* \in K$ is called a solution of the variational inequality (1.1) if, for every $y \in K$, $\langle Tx^* - \xi, y - x^* \rangle \geq 0$.

The variational inequality (1.1) has been studied by many authors (for example, see [1,2,8,9,12–14]). If B is a Hilbert space, $B^* = B$, the metric projection operator $P_K : B^*(= B) \rightarrow B$ plays a very important role in solving the variational inequality (1.1).

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In general Banach spaces, the metric projection operator $P_K : B^* \rightarrow B$ may not be defined. Alber and Guerre–Delabriers generalized the metric projection operator P_K to the so-called generalized projection operator π_K and Π_K from Hilbert spaces to uniformly convex and uniformly smooth Banach spaces. Many properties and applications of the generalized projection operator π_K and Π_K have been given in [1–7]. We list the definition and some properties of π_K and Π_K below for the easy reference. For detail one may read [1–7].

Let B be a Banach space with dual space B^* . Then the normalized duality mapping $J : B \rightarrow B^*$ is defined by taking

$$J(x) \in B^*, \quad \text{such that } \langle J(x), x \rangle = \|J(x)\| \|x\| = \|x\|^2 = \|J(x)\|^2,$$

where $\langle \cdot, \cdot \rangle$ denotes the generalized duality pairing of B^* and B . Without confusion, one understands that $\|J(x)\|$ is the B^* norm and $\|x\|$ is the B -norm. Take a functional $V : B^* \times B \rightarrow R$ is defined by the formula:

$$V(\varphi, x) = \|\varphi\|^2 - 2\langle \varphi, x \rangle + \|x\|^2,$$

where $\varphi \in B^*$ and $x \in B$.

It is easy to see that $V(\varphi, x) \geq (\|\varphi\| - \|x\|)^2$. Thus the functional $V : B^* \times B \rightarrow R^+$ is nonnegative.

Definition [1, Definition 6.2]. Operator $\pi_K : B^* \rightarrow K$ is called the generalized projection operator if it associates with an arbitrary fixed point $\varphi \in B^*$ the minimum point of the functional $V(\varphi, x)$, i.e. a solution to the minimization problem

$$V(\varphi, \pi_K \varphi) = \inf_{y \in K} V(\varphi, y). \quad (1.2)$$

$\pi_K \varphi \in K \subset B$ is then called a generalized projection of the point φ .

Applying the definitions of V and J , a functional $V_2 : B \times B \rightarrow R$ is defined by the formula:

$$V_2(x, y) = V(Jx, y), \quad \text{for all } x, y \in B.$$

The following properties of the operators J , V , and π_K are useful for our paper.

- (i) J is a monotone and bounded operator in arbitrary Banach spaces.
- (ii) J is a strictly monotone operator in strictly convex Banach spaces.
- (iii) J is a continuous operator in smooth Banach spaces.
- (iv) J is an uniformly continuous operator on each bounded set in uniformly smooth Banach spaces.
- (v) J is the identity operator in Hilbert spaces, i.e. $J = I_H$.
- (vi) $V(\varphi, x)$ is continuous.
- (vii) $V(\varphi, x)$ is convex with respect to φ when x is fixed and with respect to x when φ is fixed.
- (viii) $(\|\varphi\| - \|x\|)^2 \leq V(\varphi, x) \leq (\|\varphi\| + \|x\|)^2$.
- (ix) $V(\varphi, x) = 0$ if and only if $\varphi = Jx$.

- (x) $V(J\pi_K\varphi, x) \leq V(\varphi, x)$ for all $\varphi \in B^*$ and $x \in B$.
- (xi) The operator π_K is J fixed in each point $x \in K$, i.e. $\pi_K Jx = x$.
- (xii) π_K is monotone in B^* , i.e. for all $\varphi_1, \varphi_2 \in B^*$

$$\langle \pi_K \varphi_1 - \pi_K \varphi_2, \varphi_1 - \varphi_2 \rangle \geq 0.$$

- (xiii) If the Banach space B is uniformly smooth, then for all $\varphi_1, \varphi_2 \in B^*$, we have

$$\|\pi_K \varphi_1 - \pi_K \varphi_2\| \leq 2R_1 g_B^{-1}(\|\varphi_1 - \varphi_2\|/R_1),$$

where $R_1 = (\|\pi_K \varphi_1\|^2 + \|\pi_K \varphi_2\|^2)^{1/2}$, and g_B^{-1} is the inverse function to g_E that is defined by the modulus of smoothness for an uniformly smooth Banach space.

Once the generalized projection operator $\pi_K : B^* \rightarrow K$ is introduced, solving the variational inequality (1.1) is equivalent to finding a fixed point of a special operator from K to K . That is described by the following theorem.

Theorem A [1, Theorem 8.1]. *Let T be an arbitrary operator acting from the Banach space B to B^* , α an arbitrary fixed positive number, $\xi \in B^*$. Then the point $x^* \in K \subset B$ is a solution of the variational inequality*

$$\langle Tx - \xi, y - x \rangle = 0, \quad \text{for every } y \in K,$$

if and only if x^ is a solution of the operator equation in B*

$$x = \pi_K(Jx - \alpha(Tx - \xi)). \quad (1.3)$$

If we assume that a solution x^* of (1.1) exists, [1, Theorem 8.2], provides an iteration scheme to estimate it. In this paper, in Section 2, we study the existence of the solution of the variational inequality (1.1) by applying the above Theorem A and by using the well-known FanKKM Theorem. In Section 3, we study the approximation of the solution of the variational inequality (1.1) by a Mann type iteration scheme. We recall the FanKKM theorem below for easy reference.

Definition 1 (KKM mapping). Let K be a nonempty subset of a linear space X . A set-valued mapping $G : K \rightarrow 2^X$ is said to be a KKM mapping if for any finite subset $\{y_1, y_2, \dots, y_n\}$ of K , we have

$$\text{co}\{y_1, y_2, \dots, y_n\} \subseteq \bigcup_{i=1}^n G(y_i)$$

where $\text{co}\{y_1, y_2, \dots, y_n\}$ denotes the convex hull of $\{y_1, y_2, \dots, y_n\}$.

Theorem B (FanKKM Theorem). *Let K be a nonempty convex subset of a Hausdorff topological vector space X and let $G : K \rightarrow 2^X$ be a KKM mapping with closed values. If there exists a nonempty compact convex subset D of C such that $\bigcap_{y \in D} G(y)$ is contained in a compact subset of K , then $\bigcap_{y \in K} G(y) \neq \emptyset$.*

The FanKKM Theorem has another version.

Theorem C (FanKKM Theorem). *Let K be a nonempty convex subset of a Hausdorff topological vector space X and let $G : K \rightarrow 2^X$ be a KKM mapping with closed values. If there exists a point $y_0 \in K$ such that $G(y_0)$ is a compact subset of K , then $\bigcap_{y \in K} G(y) \neq \emptyset$.*

2. The main theorem

Theorem 2.1. *Let K be a nonempty, closed and convex subset of Banach space B with dual space B^* . Let $T : K \rightarrow B^*$ be a continuous mapping. If there exists an element $y_0 \in K$ such that the subset of K*

$$\{x \in K : 2\langle Jx - \alpha(Tx - \xi), y_0 - x \rangle + \|x\|^2 \leq \|y_0\|^2\}, \quad (2.1)$$

where $\xi \in B^$ and α is an arbitrary positive constant, is compact. Then the variational inequality (1.1) has a solution.*

Proof. From Theorem A, we only need to prove that Eq. (1.3) has a solution.

Define $G : K \rightarrow 2^K$ as follows:

$$G(y) = \{x \in K : V(Jx - \alpha(Tx - \xi), x) \leq V(Jx - \alpha(Tx - \xi), y)\}.$$

Since T is continuous, the continuity properties of V and J (iii) and (vi) yield that for every $y \in K$, $G(y)$ is nonempty and closed.

Next we prove that the map $G : K \rightarrow 2^K$ is a KKM map in K .

Let n be an arbitrary positive integer. Suppose $y_1, y_2, \dots, y_n \in K$ and $0 < \lambda_1, \lambda_2, \dots, \lambda_n \leq 1$, such that $\sum_{i=1}^n \lambda_i = 1$. Let $v = \sum_{i=1}^n \lambda_i y_i$. Applying the convexity property of V (vii), we have

$$\begin{aligned} V(Jv - \alpha(Tv - \xi), v) &= V\left(Jv - \alpha(Tv - \xi), \sum_{i=1}^n \lambda_i y_i\right) \\ &\leq \sum_{i=1}^n \lambda_i V(Jv - \alpha(Tv - \xi), y_i). \end{aligned}$$

This implies that

$$V(Jv - \alpha(Tv - \xi), v) \leq \max_{1 \leq i \leq n} V(Jv - \alpha(Tv - \xi), y_i).$$

Hence there is at least one number $j = 1, 2, \dots, n$, such that

$$V(Jv - \alpha(Tv - \xi), v) \leq V(Jv - \alpha(Tv - \xi), y_j),$$

i.e., $v \in G(y_j)$. We obtain that

$$v = \sum_{i=1}^n \lambda_i y_i \in \sum_{i=1}^n G(y_i).$$

Thus, K is a KKM mapping.

If $x \in G(y_0)$, then $V(Jx - \alpha(Tx - \xi), x) \leq V(Jx - \alpha(Tx - \xi), y_0)$. From the definition of V , we have

$$\begin{aligned} & \|Jx - \alpha(Tx - \xi)\|^2 - 2\langle Jx - \alpha(Tx - \xi), x \rangle + \|x\|^2 \\ & \leq \|Jx - \alpha(Tx - \xi)\|^2 - 2\langle Jx - \alpha(Tx - \xi), y_0 \rangle + \|y_0\|^2. \end{aligned}$$

Simplifying the above inequality, we have

$$2\langle Jx - \alpha(Tx - \xi), y_0 - x \rangle + \|x\|^2 \leq \|y_0\|^2.$$

We get that $G(y_0) = \{x \in K : 2\langle Jx - \alpha(Tx - \xi), y_0 - x \rangle + \|x\|^2 \leq \|y_0\|^2\}$. From condition (2.1), we see that $G(y_0)$ is compact. From the FanKKM Theorem, we obtain that $\bigcap_{y \in K} G(y) \neq \emptyset$. Then there exists at least one $x^* \in \bigcap_{y \in K} G(y)$; that is,

$$V(Jx^* - \alpha(Tx^* - \xi), x^*) \leq V(Jx^* - \alpha(Tx^* - \xi), y), \quad \text{for all } y \in K.$$

From the definition of the generalized projection operator $\pi_K : B^* \rightarrow K$, we obtain

$$x^* = \pi_K(Jx^* - \alpha(Tx^* - \xi)).$$

The theorem is proved. \square

Example 2.2. Let B be a Banach space with dual space B^* and K be a nonempty, closed and convex subset. For an element $y_0 \in K$ and $\xi \in B^*$, we define $T : K \rightarrow B^*$ as follows:

$$T(x) = J(x) - J(y_0 - x) - J(y_0) + \xi.$$

It is easy to see that T is continuous and T satisfies condition (2.1) in Theorem 2.1 with $\alpha = 1$. In fact,

$$\begin{aligned} & \{x \in K : 2\langle Jx - (Tx - \xi), y_0 - x \rangle + \|x\|^2 \leq \|y_0\|^2\} \\ & = \{x \in K : 2\langle J(y_0 - x) + J(y_0), y_0 - x \rangle + \|x\|^2 \leq \|y_0\|^2\} \\ & = \{x \in K : 2(\langle J(y_0 - x), y_0 - x \rangle + \langle J(y_0), y_0 - x \rangle) + \|x\|^2 \leq \|y_0\|^2\} \\ & = \{x \in K : 2(\|y_0 - x\|^2 + \|y_0\|^2 - \langle J(y_0), x \rangle) + \|x\|^2 \leq \|y_0\|^2\} \\ & = \{x \in K : 2\|y_0 - x\|^2 \leq -\|y_0\|^2 + 2\langle J(y_0), x \rangle - \|x\|^2\} \\ & \subseteq \{x \in K : 2\|y_0 - x\|^2 \leq -\|y_0\|^2 + 2\|J(y_0)\|\|x\| - \|x\|^2\} \\ & = \{x \in K : 2\|y_0 - x\|^2 \leq -\|y_0\|^2 + 2\|y_0\|\|x\| - \|x\|^2\} \\ & = \{x \in K : 2\|y_0 - x\|^2 \leq -(\|y_0\| - \|x\|)^2\} \\ & = \{y_0\}. \end{aligned}$$

From Theorem 2.1, the variational inequality (1.1) has at least one solution. As a matter of fact, y_0 is a solution of the variational inequality.

One of the most important cases is when K is a pointed convex closed cone. In this case, the origin θ of the Banach space B is in K . If we replace y_0 by θ in Theorem 2.1, we have the following corollary immediately. Since this case is important, we call it a theorem.

Theorem 2.3. Let B be a Banach space with dual space B^* and K be a nonempty, closed and convex subset that contains the origin θ of B . Let $T : K \rightarrow B^*$ be a continuous mapping. If the following subset of K is compact,

$$\left\{x \in K: \langle Tx - \xi, x \rangle \leq \frac{1}{2\alpha} \|x\|^2\right\}, \quad (2.2)$$

where $\xi \in B^*$ and α is an arbitrary positive constant, then the variational inequality (1.1) has a solution.

Proof. Taking $y_0 = \theta$ in condition (2.1) and noticing $\langle Jx, x \rangle = \|x\|^2$, we have the condition (2.2). The theorem is proved. \square

The following corollary follows from Theorem 2.3 immediately.

Corollary 2.4. Let B be a Banach space with dual space B^* and let K be a pointed convex closed cone. Let $T: K \rightarrow B^*$ be a continuous mapping. If the following subset of K is compact,

$$\left\{x \in K: \langle Tx - \xi, x \rangle \leq \frac{1}{2\alpha} \|x\|^2\right\},$$

where $\xi \in B^*$ and α is an arbitrary positive constant, then the variational inequality (1.1) has a solution.

Remark. It is interesting to note that in Theorems 2.1, 2.3, and Corollary 2.4, the positive constant α is arbitrary. If $\alpha_1 > \alpha_2 > 0$, then we have

$$\left\{x \in K: \langle Tx - \xi, x \rangle \leq \frac{1}{2\alpha_1} \|x\|^2\right\} \subseteq \left\{x \in K: \langle Tx - \xi, x \rangle \leq \frac{1}{2\alpha_2} \|x\|^2\right\}.$$

Hence

$$\bigcap_{\alpha > 0} \left\{x \in K: \langle Tx - \xi, x \rangle \leq \frac{1}{2\alpha} \|x\|^2\right\} = \{x \in K: \langle Tx - \xi, x \rangle \leq 0\}.$$

The next theorem describes the solution set of the variational inequality (1.1).

Theorem 2.5. Let K be a nonempty, closed and convex subset of a Banach space B with dual space B^* . Let $T: K \rightarrow B^*$ be a continuous mapping. If there exists an element $y_0 \in K$ such that the subset of K

$$\{x \in K: \|x\| - 2\|Jx - \alpha(Tx - \xi)\| \leq \|y_0\|\}, \quad (2.3)$$

where $\xi \in B^*$ and α is an arbitrary positive constant, is compact, then the solution set of the variational inequality (1.1) is not empty and is given by

$$\bigcap_{\alpha > 0} \bigcap_{y \in K} \{x \in K: V(Jx - \alpha(Tx - \xi), x) \leq V(Jx - \alpha(Tx - \xi), y)\}.$$

If K contains the origin θ of B , then condition (2.1) can be replaced by condition (2.2).

Proof. From the definition of V , we have

$$\begin{aligned}
 & \bigcap_{\alpha > 0} \bigcap_{y \in K} \{x \in K: V(Jx - \alpha(Tx - \xi), x) \leq V(Jx - \alpha(Tx - \xi), y)\} \\
 &= \bigcap_{\alpha > 0} \bigcap_{y \in K} \{x \in K: -2\langle Jx - \alpha(Tx - \xi), x \rangle + \|x\|^2 \\
 &\quad \leq -2\langle Jx - \alpha(Tx - \xi), y \rangle + \|y\|^2\} \\
 &= \bigcap_{\alpha > 0} \bigcap_{y \in K} \{x \in K: 2\alpha\langle Tx - \xi, x - y \rangle - \|x\|^2 \leq -2\langle Jx, y \rangle + \|y\|^2\} \\
 &= \bigcap_{\alpha > 0} \bigcap_{y \in K} \{x \in K: 2\alpha\langle Tx - \xi, x - y \rangle \leq \|x\|^2 - 2\langle Jx, y \rangle + \|y\|^2\}.
 \end{aligned}$$

Since $\langle Jx, y \rangle \leq \|Jx\| \|y\| \leq \|x\| \|y\|$, we see that $\|x\|^2 - 2\langle Jx, y \rangle + \|y\|^2 \geq \|x\|^2 - 2\|x\| \|y\| + \|y\|^2 = (\|x\| - \|y\|)^2 \geq 0$. This yields that $2\alpha\langle Tx - \xi, x - y \rangle \leq \|x\|^2 - 2\langle Jx, y \rangle + \|y\|^2$, for all $\alpha > 0$, if and only if $\langle Tx - \xi, x - y \rangle \leq 0$. We obtain

$$\begin{aligned}
 & \bigcap_{\alpha > 0} \bigcap_{y \in K} \{x \in K: 2\alpha\langle Tx - \xi, x - y \rangle \leq \|x\|^2 - 2\langle Jx, y \rangle + \|y\|^2\} \\
 &= \bigcap_{y \in K} \{x \in K: \langle Tx - \xi, x - y \rangle \leq 0\} \\
 &= \bigcap_{y \in K} \{x \in K: \langle Tx - \xi, y - x \rangle \geq 0\}.
 \end{aligned}$$

The theorem is proved. \square

3. An approximation of the solutions of the variational inequality (1.1)

Suppose that the variational inequality (1.1) has a (unique) solution x^* . For any $x_0 \in K$, we define the following successive sequence:

$$x_{n+1} = \pi_K(Jx_n - \alpha_n(Tx_n - \xi)), \quad n = 1, 2, 3, \dots$$

In [1, Theorem 8.2], Alber proved that the above sequence converges to the solution x^* , i.e. $\|x_n - x^*\| \rightarrow 0$, as $n \rightarrow \infty$, if the following conditions hold:

- (i) B is a uniformly convex and uniformly smooth Banach space;
- (ii) $T: B \rightarrow B^*$ is uniformly monotone, i.e.,

$$\langle Tx_1 - Tx_2, x_1 - x_2 \rangle \geq \psi(\|x_1 - x_2\|), \quad \text{for all } x_1, x_2 \in B,$$

where $\psi(t)$ is a continuous strictly increasing function for all $t \geq 0$ with $\psi(0) = 0$;

- (iii) $T: B \rightarrow B^*$ has φ arbitrary growth, i.e.,

$$\|Tx - \xi\| \leq \varphi(\|x - x^*\|), \quad \text{for all } x \in B,$$

where $\varphi(t)$ is a continuous nondecreasing function for all $t \geq 0$ with $\varphi(0) \geq 0$.

In this section, we study the approximation of the solution of the variational inequality (1.1) by a Mann sequence. The conditions for T are different from the above three conditions. The techniques used in this section have been used by many authors (see [15–19]). The following lemma given by Chidume and Li in [10] is useful for the proof of the theorem in this section.

Lemma 3.1 [10, Lemma 3.4]. *Let B be a uniformly convex Banach space. Then for arbitrary $r > 0$, there exists a continuous, strictly increasing convex function $g : \mathbb{R}^+ \rightarrow \mathbb{R}^+$, $g(0) = 0$, such that for all x_1, x_2 and $y \in B_r(0) := \{x \in B : \|x\| \leq r\}$ and for any $\alpha \in [0, 1]$, the following inequality holds:*

$$V_2(\alpha x_1 + (1 - \alpha)x_2, y) \leq \alpha V_2(x_1, y) + (1 - \alpha)V_2(x_2, y) - \alpha(1 - \alpha)g(\|x_1 - x_2\|). \quad (3.1)$$

Theorem 3.1. *Let B be an uniformly convex and uniformly smooth Banach space and let K be a compact convex subset of B . Let $T : K \rightarrow B^*$ be a continuous mapping on K such that*

$$\langle Tx - \xi, J^*(Jx - (Tx - \xi)) \rangle \geq 0, \quad \text{for all } x \in K, \quad (3.2)$$

where $\xi \in B^*$. For any $x_0 \in K$, we define a Mann type iteration scheme as follows:

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n\pi_K(Jx_n - (Tx_n - \xi)), \quad n = 1, 2, 3, \dots, \quad (3.3)$$

where $\{\alpha_n\}$ satisfies conditions

- (a) $0 \leq \alpha_n \leq 1$, for all n ,
- (b) $\sum_{n=1}^{\infty} \alpha_n(1 - \alpha_n) = \infty$.

Then the variational inequality (1.1) has a solution $x^* \in K$ and there exists a subsequence $\{n(i)\} \subseteq \{n\}$ such that

$$x_{n(i)} \rightarrow x^*, \quad \text{as } i \rightarrow \infty.$$

Proof. For any $y \in B$, from definition (3.3) and inequality (3.3) of Lemma 3.1, we have

$$V_2(x_{n+1}, y) \leq (1 - \alpha_n)V_2(x_n, y) + \alpha_n V_2(\pi_K(Jx_n - (Tx_n - \xi)), y) - \alpha_n(1 - \alpha_n)g(\|\pi_K(Jx_n - (Tx_n - \xi)) - x_n\|). \quad (3.4)$$

From the definition of the functional V_2 , and the convexity property of the Lyapunov functional V , we obtain

$$\begin{aligned} V_2(\pi_K(Jx_n - (Tx_n - \xi)), y) &= V(J\pi_K(Jx_n - (Tx_n - \xi)), y) \leq V(Jx_n - (Tx_n - \xi), y) \\ &\leq V(Jx_n, y) + 2\langle -(Tx_n - \xi), J^*(Jx_n - (Tx_n - \xi)) - y \rangle \\ &= V_2(x_n, y) - 2\langle (Tx_n - \xi), J^*(Jx_n - (Tx_n - \xi)) - y \rangle. \end{aligned}$$

Substituting the above inequality into inequality (3.4) and applying condition (3.1) we have

$$\begin{aligned}
V_2(x_{n+1}, y) &\leq (1 - \alpha_n)V_2(x_n, y) + \alpha_n(V_2(x_n, y) \\
&\quad - 2\alpha_n\langle Tx_n - \xi, J^*(Jx_n - (Tx_n - \xi)) - y \rangle) \\
&\quad - \alpha_n(1 - \alpha_n)g(\|\pi_K(Jx_n - (Tx_n - \xi)) - x_n\|) \\
&\leq V_2(x_n, y) + 2\alpha_n\langle Tx_n - \xi, y \rangle \\
&\quad - \alpha_n(1 - \alpha_n)g(\|\pi_K(Jx_n - (Tx_n - \xi)) - x_n\|).
\end{aligned}$$

Taking $y = \theta$ in the above inequality and taking the sum for $i = 1, 2, \dots, n$, we obtain

$$\sum_{i=1}^n \alpha_i(1 - \alpha_i)g(\|\pi_K(Jx_i - (Tx_i - \xi)) - x_i\|) \leq V_2(x_0, \theta) - V_2(x_{n+1}, \theta).$$

Since $V_2: B \times B \rightarrow R^+$ is nonnegative and $V_2(x_0, \theta) < \infty$, the sum becomes

$$\sum_{i=1}^{\infty} \alpha_i(1 - \alpha_i)g(\|\pi_K(Jx_i - (Tx_i - \xi)) - x_i\|) < \infty.$$

From the condition $\sum_{i=1}^{\infty} \alpha_i(1 - \alpha_i) = \infty$, the above inequality insures that there exist a subsequence $\{n(i)\} \subseteq \{n\}$ such that

$$g(\|\pi_K(Jx_{n(i)} - (Tx_{n(i)} - \xi)) - x_{n(i)}\|) \rightarrow 0, \quad \text{as } i \rightarrow \infty.$$

Applying the properties of g , we have

$$\|\pi_K(Jx_{n(i)} - (Tx_{n(i)} - \xi)) - x_{n(i)}\| \rightarrow 0, \quad \text{as } i \rightarrow \infty. \quad (3.5)$$

From the compactness of K , the sequence $\{x_{n(i)}\}$ must have a subsequence that converges to a point $x^* \in K$. Without loss of the generality, we may assume that the subsequence is $\{x_{n(i)}\}$, that is,

$$x_{n(i)} \rightarrow x^*, \quad \text{as } i \rightarrow \infty. \quad (3.6)$$

Using the continuity properties of the operators π , J and T and combining (3.5) and (3.6) we obtain

$$\pi_K(Jx^* - (Tx^* - \xi)) = x^*.$$

From Theorem A, the above equality implies that x^* is a solution of the variational inequality (1.1). The theorem is proved. \square

Theorem 3.2. *Let B be an uniformly convex and uniformly smooth Banach space and let K be a compact convex subset of B . Let $T: K \rightarrow B^*$ be a continuous mapping on K . Suppose that B and T satisfy the following conditions*

- (i) $g_b^{-1}(t) \leq t/2$, for all $t \geq 0$, where g_B^{-1} is the inverse function to g_E that is defined by the modulus of smoothness for the uniformly smooth Banach spaces B .
- (ii) $\langle Tx - \xi, J^*(Jx - (Tx - \xi)) \rangle \geq 0$, for all $x \in K$ and some $\xi \in B^*$.
- (iii) $J - T$ is nonexpansive.

Then the variational inequality (1.1) has a solution $x^* \in K$ and for any $x_0 \in K$, the Mann sequence (3.3) converges to x^* , that is,

$$x_n \rightarrow x^*, \quad \text{as } n \rightarrow \infty.$$

Proof. From Theorem 3.1 and condition (ii) in this theorem, we see that the variational inequality (1.1) has a solution $x^* \in K$ and x^* satisfies Eq. (1.3).

Since α is an arbitrary positive number, we may take $\alpha = 1$. Then the above property can be restated as follows: the variational inequality (1.1) has a solution $x^* \in K$ and x^* satisfies the following equation:

$$x^* = \pi_K(x^* - (Tx^* - \xi)).$$

Applying property (xiii) and condition (i) and (iii) of Theorem 3.2, we have

$$\begin{aligned} \|x_{n+1} - x^*\| &= \|(1 - \alpha_n)x_n + \alpha_n \pi_K(x_n - (Tx_n - \xi)) - x^*\| \\ &= \|(1 - \alpha_n)(x_n - x^*) + \alpha_n(\pi_K(x_n - (Tx_n - \xi)) - x^*)\| \\ &\leq (1 - \alpha_n)\|x_n - x^*\| + \alpha_n\|\pi_K(x_n - (Tx_n - \xi)) - x^*\| \\ &\leq (1 - \alpha_n)\|x_n - x^*\| \\ &\quad + \alpha_n\|\pi_K(x_n - (Tx_n - \xi)) - \pi_K(x^* - (Tx^* - \xi))\| \\ &\leq (1 - \alpha_n)\|x_n - x^*\| \\ &\quad + \alpha_n 2R_1 g_B^{-1}(\|(x_n - (Tx_n - \xi)) - (x^* - (Tx^* - \xi))\|/R_1) \\ &= (1 - \alpha_n)\|x_n - x^*\| + \alpha_n\|(I - T)x_n - (I - T)x^*\| \\ &\leq \|x_n - x^*\|, \end{aligned}$$

where $R_1 = (\|\pi_K(x_n - (Tx_n - \xi))\|^2 + \|\pi_K(x^* - (Tx^* - \xi))\|^2)^{1/2}$.

We see that the sequence $\{\|x_n - x^*\|\}$ is a decreasing sequence. From Theorem 3.1, it has a subsequence that converges to 0. Thus the sequence converges to 0. The theorem is proved. \square

If $B = H$ is a Hilbert space, then $\pi_K = P_K$, $H^* = H$, $J^* = J = I_K$, and we have the following corollary immediate.

Corollary 3.3. Let H be a Hilbert space and let K be a compact convex subset of H . Let $T : K \rightarrow H$ be a continuous mapping on K such that

$$\langle Tx - \xi, x - (Tx - \xi) \rangle \geq 0, \quad \text{for all } x \in K, \quad (3.2)$$

where $\xi \in H$. For any $x_0 \in K$, we define a Mann type iteration scheme as follows:

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n P_K(x_n - (Tx_n - \xi)), \quad n = 1, 2, 3, \dots, \quad (3.3)$$

where $\{\alpha_n\}$ satisfies conditions

- (a) $0 \leq \alpha_n \leq 1$, for all n ,
- (b) $\sum_{i=1}^{\infty} \alpha_n(1 - \alpha_n) = \infty$.

Then the variational inequality (1.1) has a solution $x^* \in K$ and there exists a subsequence $\{n(i)\} \subseteq \{n\}$ such that

$$x_{n(i)} \rightarrow x^*, \quad \text{as } i \rightarrow \infty.$$

If H is a Hilbert space, then $\pi_K = P_K$, is the projection operator. It is well known that the projection operator P_K is nonexpansive. Then we have the following result.

Corollary 3.4. *Let H be a Hilbert space and let K be a compact convex subset of H . Let $T : K \rightarrow H$ be a continuous mapping on K . If T satisfies the following conditions:*

- (i) $\langle Tx - \xi, x - (Tx - \xi) \rangle \geq 0$, for all $x \in K$, and $\xi \in H$,
- (ii) $I - T$ is nonexpansive.

Then the variational inequality (1.1) has a solution $x^* \in K$ and

$$x_n \rightarrow x^*, \quad \text{as } n \rightarrow \infty,$$

where $\{x_n\}$ is defined by (3.3) and for any $x_0 \in K$.

Proof. The proof of the corollary is almost the same as the proof of Theorem 3.2.

From Corollary 3.3 and condition (i) in this theorem, we see that the variational inequality (1.1) has a solution $x^* \in K$ and x^* satisfies Eq. (1.3) with $\alpha = 1$.

Applying the above equality and the nonexpansive property of the operators P_K and $I - T$, we have

$$\begin{aligned} \|x_{n+1} - x^*\| &= \|(1 - \alpha_n)x_n + \alpha_n P_K(x_n - (Tx_n - \xi)) - x^*\| \\ &\leq (1 - \alpha_n)\|x_n - x^*\| \\ &\quad + \alpha_n \|P_K(x_n - (Tx_n - \xi)) - P_K(x^* - (Tx^* - \xi))\| \\ &\leq (1 - \alpha_n)\|x_n - x^*\| + \alpha_n \|x_n - (Tx_n - \xi) - (x^* - (Tx^* - \xi))\| \\ &= (1 - \alpha_n)\|x_n - x^*\| + \alpha_n \|(I - T)x_n - (I - T)x^*\| \\ &\leq \|x_n - x^*\|. \end{aligned}$$

The rest of the proof is same as the proof of Theorem 3.2. The corollary is proved. \square

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